

A NOTE ON THE FOURIER COEFFICIENTS AND PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to investigate Paley type and Hardy-Littlewood type inequalities and strong convergence theorem of partial sums of Vilenkin-Fourier series.

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Let \mathbb{N}_+ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive numbers, not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k .

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j} 's.

The direct product μ , of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad (j \in Z_{m_k})$$

is the Haar measure on G_m , with $\mu(G_m) = 1$.

If $\sup_n m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded then G_m is said to be an unbounded Vilenkin group. **In this paper we discuss bounded Vilenkin groups only.**

The elements of G_m represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m, n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\bar{I}_n := G_m \setminus I_n$.

If we define the so-called generalized number system, based on m in the following way :

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ ($j \in \mathbb{N}$) and only a finite number of n_j 's differ from zero.

Let $|n| := \max \{j \in \mathbb{N} : n_j \neq 0\}$.

Denote by \mathbb{N}_{n_0} the subset of positive integers \mathbb{N}_+ , for which $n_{|n|} = n_0 = 1$. Then for every $n \in \mathbb{N}_{n_0}$, $M_k < n < M_{k+1}$ can be written as $n = M_0 + \sum_{j=1}^{k-1} n_j M_j + M_k = 1 + \sum_{j=1}^{k-1} n_j M_j + M_k$, where $n_j \in \{0, m_j - 1\}$, ($j \in \mathbb{N}_+$).

By simple calculation we get

$$(1) \quad \sum_{\{n: M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{n_0}\}} 1 = \frac{M_{k-1}}{m_0} \geq c M_k,$$

where c is absolute constant.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space.

Next, we introduce on G_m an ortonormal system, which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \rightarrow \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k), \quad (\iota^2 = -1, x \in G_m, k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is ortonormal and complete in $L_2(G_m)$ [1, 13].

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the the Fourier coefficients, the partial sums, the Dirichlet kernels, with respect to the Vilenkin system in the usual manner:

$$\begin{aligned} \widehat{f}(k) &: = \int_{G_m} f \bar{\psi}_k d\mu & (k \in \mathbb{N}), \\ S_n f &: = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k & (n \in \mathbb{N}_+, S_0 f := 0), \\ D_n &: = \sum_{k=0}^{n-1} \psi_k & (n \in \mathbb{N}_+), \end{aligned}$$

Recall that

$$(2) \quad D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n \\ 0 & \text{if } x \notin I_n \end{cases}$$

and

$$(3) \quad D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u \right).$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p d\mu \right)^{1/p} \quad (0 < p < \infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty.$$

The σ -algebra, generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n ($n \in \mathbb{N}$). The conditional expectation operators relative to F_n ($n \in \mathbb{N}$) are denoted by E_n . Then

$$\begin{aligned} E_n f(x) &= S_{M_n} f(x) = \sum_{k=0}^{M_n-1} \widehat{f}(k) w_k \\ &= \frac{1}{|I_n(x)|} \int_{I_n(x)} f(x) d\mu(x), \end{aligned}$$

where $|I_n(x)| = M_n^{-1}$ denotes the length of $I_n(x)$.

A sequence $f = (f^{(n)}, n \in \mathbb{N})$ of functions $f_n \in L_1(G)$ is said to be a dyadic martingale if (for details see e.g. [14])

- (i) $f^{(n)}$ is F_n measurable, for all $n \in \mathbb{N}$,
- (ii) $E_n f^{(m)} = f^{(n)}$, for all $n \leq m$.

The maximal function of a martingale f is denoted by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In case $f \in L_1$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1$, then it is easy to show that the sequence $(S_{M_n} f : n \in \mathbb{N})$ is a martingale. If $f = (f^{(n)}, n \in \mathbb{N})$ is martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$(4) \quad \widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)}(x) \overline{\Psi}_i(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from f .

A bounded measurable function a is p -atom, if there exist a dyadic interval I , such that

$$\begin{cases} a) & \int_I a d\mu = 0 \\ b) & \|a\|_\infty \leq \mu(I)^{-1/p} \\ c) & \text{supp}(a) \subset I. \end{cases}$$

The Hardy martingale spaces $H_p(G_m)$, for $0 < p \leq 1$ have an atomic characterization. Namely, the following theorem is true (see [15]):

Theorem W. A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in H_p ($0 < p \leq 1$) if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p -atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers, such that for every $n \in \mathbb{N}$:

$$(5) \quad \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)},$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, $\|f\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p}$, where the infimum is taken over all decomposition of f of the form (5).

When $0 < p \leq 1$, the Hardy martingale space H_p is proper subspace of Lebesgue space L_p . It is well known that for $1 < p < \infty$ the space H_p is nothing but L_p .

The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis. Namely,

$$\sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|}{k} \leq c \|f\|_{H_1},$$

where the function f belongs to the Hardy space H_1 and c is an absolute constant. This was proved in the trigonometric case by Hardy and Littlewood [6] (see also Coifman and Weiss [2]) and for Walsh system it can be found in [8].

Weisz [14, 16] generalized this result for Vilenkin system and proved:

Theorem A. Let $0 < p \leq 2$. Then there is an absolute constant c_p , depend only p , such that

$$(6) \quad \sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^p}{k^{2-p}} \leq c_p \|f\|_{H_p},$$

for all $f \in H_p$.

Paley [7] proved that the Walsh- Fourier coefficients of a function $f \in L_p$ ($1 < p < 2$) satisfy the condition

$$\sum_{k=1}^{\infty} |\widehat{f}(2^k)|^2 < \infty.$$

This results fails to hold for $p = 1$. However, it can be verified for functions $f \in L_1$, such that f^* belongs L_1 . i.e $f \in H_1$ (see e.g Coifman and Weiss [2]).

For the Vilenkin system the following theorem (see Weisz [17]) is proved:

Theorem B. Let $0 < p \leq 1$. Then there is an absolute constant c_p , depend only p , such that

$$(7) \quad \left(\sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \widehat{f}(jM_k) \right|^2 \right)^{1/2} \leq c_p \|f\|_{H_p},$$

for all $f \in H_p$.

It is well-known that Vilenkin system forms not basis in the space L_1 . Moreover, there is a function in the dyadic Hardy space H_1 , such that the partial sums of f are not bounded in L_1 -norm. However, in Simon [10] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the k -th partial sum of the Walsh-Fourier series of f . (For the trigonometric analogue see Smith [11], for the Vilenkin system by Gát [3]). For the Vilenkin system Simon [9] proved:

Theorem C. Let $0 < p < 1$. Then there is an absolute constant c_p , depends only p , such that

$$(8) \quad \sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p,$$

for all $f \in H_p$.

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [18], Goginava [5], Gogoladze [4], Tephnadze [12].

The main aim of this paper is to prove that the following is true:

Theorem 1. Let $\{\Phi_n\}_{n=1}^{\infty}$ is any nondecreasing sequence, satisfying the condition $\lim_{n \rightarrow \infty} \Phi_n = +\infty$. Then there exists a martingale $f \in H_p$, such that

$$(9) \quad \sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^p \Phi_k}{k^{2-p}} = \infty, \quad \text{for } 0 < p \leq 2,$$

$$(10) \quad \sum_{k=1}^{\infty} \frac{\Phi_{M_k}}{M_k^{2/p-2}} \sum_{j=1}^{m_k-1} \left| \widehat{f}(jM_k) \right|^2 = \infty, \quad \text{for } 0 < p \leq 1$$

and

$$(11) \quad \sum_{k=1}^{\infty} \frac{\|S_k f\|_{L_{p,\infty}}^p \Phi_k}{k^{2-p}} = \infty, \quad \text{for } 0 < p < 1.$$

Proof of Theorem 1. Let $0 < p \leq 2$ and $\{\Phi_n\}_{n=1}^\infty$ is any nondecreasing, nonnegative sequence, satisfying condition

$$\lim_{n \rightarrow \infty} \Phi_n = \infty.$$

For this function $\Phi(n)$, there exists an increasing sequence $\{\alpha_k \geq 2 : k \in \mathbb{N}_+\}$ of the positive integers such that:

$$(12) \quad \sum_{k=1}^{\infty} \frac{1}{\Phi_{M_{\alpha_k}}^{p/4}} < \infty.$$

Let

$$f^{(A)}(x) := \sum_{\{k; \alpha_k < A\}} \lambda_k a_k(x),$$

where

$$\lambda_k = \frac{1}{\Phi_{M_{\alpha_k}}^{1/4}}$$

and

$$a_k(x) = \frac{M_{\alpha_k}^{1/p-1}}{M} \left(D_{M_{\alpha_k+1}}(x) - D_{M_{\alpha_k}}(x) \right),$$

where $M = \sup_{n \in \mathbb{N}} m_n$.

It is easy to show that the martingale $f = (f^{(1)}, f^{(2)}, \dots, f^{(A)}, \dots) \in H_p$.

Indeed, since

$$(13) \quad S_{M_A}(a_k(x)) = \begin{cases} a_k(x) & \alpha_k < A \\ 0 & \alpha_k \geq A, \end{cases}$$

$$\begin{aligned} \text{supp}(a_k) &= I_{\alpha_k}, \\ \int_{I_{\alpha_k}} a_k d\mu &= 0, \end{aligned}$$

and

$$\begin{aligned} \|a_k\|_\infty &\leq \frac{M_{\alpha_k}^{1/p-1}}{M} M_{\alpha_k+1} \\ &\leq M_{\alpha_k}^{1/p} = \mu(\text{supp } a_k)^{-1/p}, \end{aligned}$$

if we apply Theorem W and (12) we conclude that $f \in H_p$.

It is easy to show that

$$(14) \quad \widehat{f}(j) = \begin{cases} \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, & \text{if } j \in \{M_{\alpha_k}, \dots, M_{\alpha_k+1} - 1\}, \quad k = 1, 2, \dots \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha_k}, \dots, M_{\alpha_k+1} - 1\}. \end{cases}$$

First we prove equality (9). Using (14) we can

$$\begin{aligned} & \sum_{l=1}^{M_{\alpha_k+1}-1} \frac{|\widehat{f}(l)|^p \Phi_l}{l^{2-p}} \\ &= \sum_{n=1}^k \sum_{l=M_{\alpha_n}}^{M_{\alpha_{n+1}}-1} \frac{|\widehat{f}(l)|^p \Phi_l}{l^{2-p}} \\ &\geq \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{|\widehat{f}(l)|^p \Phi_l}{l^{2-p}} \\ &\geq c \Phi_{M_{\alpha_k}} \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{|\widehat{f}(l)|^p}{l^{2-p}} \\ &\geq c \Phi_{M_{\alpha_k}} \frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}} \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{1}{l^{2-p}} \\ &\geq c \Phi_{M_{\alpha_k}}^{1/2} M_{\alpha_k}^{1-p} \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{1}{M_{\alpha_k+1}^{2-p}} \\ &\geq c \Phi_{M_{\alpha_k}}^{1/2} M_{\alpha_k}^{1-p} \frac{1}{M_{\alpha_k+1}^{1-p}} \\ &\geq c \Phi_{M_{\alpha_k}}^{1/2} \rightarrow \infty, \quad \text{when } k \rightarrow \infty. \end{aligned}$$

Next we prove equality (10). Let $0 < p \leq 1$. Using (14) we get

$$\begin{aligned} & \sum_{l=1}^k M_{\alpha_l}^{2-2/p} \Phi_{M_{\alpha_l}} \sum_{j=1}^{m_{\alpha_l}-1} |\widehat{f}(j M_{\alpha_l})|^2 \\ &\geq M_{\alpha_k}^{2-2/p} \Phi_{M_{\alpha_k}} \sum_{j=1}^{m_{\alpha_k}-1} |\widehat{f}(j M_{\alpha_k})|^2 \\ &\geq c M_{\alpha_k}^{2-2/p} \Phi_{M_{\alpha_k}} \sum_{j=1}^{m_{\alpha_k}-1} \frac{M_{\alpha_k}^{2/p-2}}{\Phi_{M_{\alpha_k}}^{1/2}} \\ &\geq c \Phi_{M_{\alpha_k}}^{1/2} \rightarrow \infty, \quad \text{when } k \rightarrow \infty. \end{aligned}$$

Finally we prove equality (11). Let $0 < p < 1$ and $M_{\alpha_k} \leq j < M_{\alpha_k+1}$. From (14) we have

$$\begin{aligned}
S_j f(x) &= \sum_{l=0}^{M_{\alpha_{k-1}+1}-1} \widehat{f}(l) \psi_l(x) \\
&+ \sum_{l=M_{\alpha_k}}^{j-1} \widehat{f}(l) \psi_l(x) \\
&= \sum_{\eta=0}^{k-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \widehat{f}(v) \psi_v(x) \\
&+ \sum_{v=M_{\alpha_k}}^{j-1} \widehat{f}(v) \psi_v(x) \\
&= \sum_{\eta=0}^{k-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \frac{1}{M} \frac{M_{\alpha_\eta}^{1/p-1}}{\Phi_{M_{\alpha_\eta}}^{1/4}} \psi_v(x) \\
&+ \sum_{v=M_{\alpha_k}}^{j-1} \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \psi_v(x) \\
&= \sum_{\eta=0}^{k-1} \frac{1}{M} \frac{M_{\alpha_\eta}^{1/p-1}}{\Phi_{M_{\alpha_\eta}}^{1/4}} \left(D_{M_{\alpha_\eta+1}}(x) - D_{M_{\alpha_\eta}}(x) \right) \\
&+ \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left(D_j(x) - D_{M_{\alpha_k}}(x) \right) \\
&= I + II.
\end{aligned}$$

Let $j \in \mathbb{N}_{n_0}$ and $x \in G_m \setminus I_1$. Since $j - M_{\alpha_k} \in \mathbb{N}_{n_0}$ and

$$D_{j+M_{\alpha_k}}(x) = D_{M_{\alpha_k}}(x) + \psi_{M_{\alpha_k}}(x) D_j(x), \text{ when } j < M_{\alpha_k},$$

combining (2) and (3) we can write

$$\begin{aligned}
(15) \quad |II| &= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left| \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}}(x) \right| \\
&= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left| \psi_{M_{\alpha_k}}(x) \psi_{j-M_{\alpha_k}}(x) r_0^{m_0-1}(x) D_1(x) \right| \\
&= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}.
\end{aligned}$$

Applying (2) and condition $\alpha_n \geq 2$ ($n \in \mathbb{N}$) for I we have

$$(16) \quad I = 0, \quad \text{for } x \in G_m \setminus I_1.$$

It follows that

$$|S_j f(x)| = |II| = \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, \quad \text{for } x \in G_m \setminus I_1.$$

Hence

$$(17) \quad \begin{aligned} & \|S_j(f(x))\|_{L_{p,\infty}} \\ & \geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left(x \in G_m : |S_j(f(x))| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p} \\ & \geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left(x \in G_m \setminus I_1 : |S_j(f(x))| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p} \\ & = \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} |G_m \setminus I_1| \\ & \geq \frac{cM_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}. \end{aligned}$$

Combining (1) and (17) we have

$$\begin{aligned} & \sum_{j=1}^{M_{\alpha_k+1}-1} \frac{\|S_j(f(x))\|_{L_{p,\infty}}^p \Phi_j}{j^{2-p}} \\ & \geq \sum_{j=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{\|S_j(f(x))\|_{L_{p,\infty}}^p \Phi_j}{j^{2-p}} \\ & \geq \Phi_{M_{\alpha_k}} \sum_{\{n: M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{n_0}\}} \frac{\|S_j(f(x))\|_{L_{p,\infty}}^p}{j^{2-p}} \\ & \geq c \Phi_{M_{\alpha_k}} \frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}} \sum_{\{n: M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{n_0}\}} \frac{1}{j^{2-p}} \\ & \geq c \Phi_{M_{\alpha_k}}^{3/4} M_{\alpha_k}^{1-p} \sum_{\{n: M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{n_0}\}} \frac{1}{M_{\alpha_k+1}^{2-p}} \end{aligned}$$

$$\begin{aligned}
&\geq c \frac{\Phi_{M_{\alpha_k}}^{3/4}}{M_{\alpha_k+1}} \sum_{\{n: M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{n_0}\}} 1 \\
&\geq c \Phi_{M_{\alpha_k}}^{3/4} \rightarrow \infty, \quad \text{when } k \rightarrow \infty.
\end{aligned}$$

Theorem 1 is proved.

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